Journal of Global Optimization **30:** 335–346, 2004.

The Complexity of Minimum Ratio Spanning Tree Problems

Christopher C. Skiścim[1] and Susan W. Palocsay[2]
[1]*Lion Technologies, Inc., 1921 Thurston Road, Comus, MD 20842, USA (chskiscim@earthlink.net)*
[2]*Information Technology and Management Science Program, James Madison University, Harrisonburg, VA 22087, USA (palocssw@jmu.edu, www.jmu.edu)*

(Received: 4 February 2002; accepted: 20 March 2003)

Abstract. We examine the complexity of two minimum spanning tree problems with rational objective functions. We show that the Minimum Ratio Spanning Tree problem is NP-hard when the denominator is unrestricted in sign, thereby sharpening a previous complexity result. We then consider an extension of this problem where the objective function is the sum of two linear ratios whose numerators and denominators are strictly positive. This problem is shown to be NP-hard as well. We conclude with some results characterizing sufficient conditions for a globally optimal solution.

Key words: Combinatorial optimization, Fractional programming, NP-hard.

1. Introduction

The Minimum Ratio Spanning Tree (MRST) problem was introduced by Chandrasekaran [2]. It is the problem of finding a minimum spanning tree when the objective function is the ratio of two linear cost functions (e.g., a ratio of cost to weight). Chandrasekaran demonstrated the failure of the greedy algorithm to solve this problem and gave a polynomial time algorithm for solving the MRST under the restriction that the denominator remained strictly positive. The algorithm is essentially Newton's algorithm as articulated in [3]. If the denominator takes on negative values, he showed that the problem became NP-complete.

More recently, Skiscim and Palocsay [7] considered an extension of the MRST problem wherein the objective function was the sum of two ratios, each comprised of linear functions. They presented an algorithm to find the global optimal solution to this problem by adapting the algorithm of Falk and Palocsay [5]. The algorithm required the solution of NP-hard subproblems. Based on this it was conjectured that the Two Ratio Minimum Spanning Tree problem (TRMST) was also NP-hard.

In this paper we demonstrate that the MRST is NP-hard when the denominator takes on negative values thereby sharpening the earlier complexity result. We also show that TRMST is NP-hard when the numerator and denominator of both ratios are strictly positive. The paper concludes with some results concerning conditions for an optimal solution and attempts to provide some insight into what makes the TRMST so difficult.

2. Preliminaries

The MRST problem is a network optimization problem defined on an undirected network $\mathcal{G}(\mathcal{V},\mathcal{A})$ where $\mathcal{V}$ denotes the set of vertices or nodes and $\mathcal{A}$ denotes the set of edges or arcs connecting the nodes. Each arc $i\in\mathcal{A}$ has a cost $a_i>0$ and a weight $b_i>0$. Let $\mathfrak{T}$ denote the set of spanning trees of $\mathcal{G}(\mathcal{V},\mathcal{A})$. The first problem, termed $\mathcal{P}_1$, is to find a subset of arcs forming a spanning tree $\mathcal{T}\in\mathfrak{T}$ such that

$$z=\min_{\mathcal{T}\in\mathfrak{T}}\frac{\sum_{i\in\mathcal{T}}a_i}{\sum_{i\in\mathcal{T}}b_i}. \tag{1}$$

The TRMST is similarly defined. In addition to the cost and weight $a_i>0$ and $b_i>0$, we have the cost and weights $c_i>0$ and $d_i>0$ for all $i\in\mathcal{A}$, respectively. Again, the problem is to find a subset of arcs forming a spanning tree $\mathcal{T}\in\mathfrak{T}$ such that

$$z=\min_{\mathcal{T}\in\mathfrak{T}}\frac{\sum_{i\in\mathcal{T}}a_i}{\sum_{i\in\mathcal{T}}b_i}+\frac{\sum_{i\in\mathcal{T}}c_i}{\sum_{i\in\mathcal{T}}d_i} \tag{2}$$

We call this problem $\mathcal{P}_2$. Throughout, we make use of the following NP-complete problem BIPARTITION:

INSTANCE A finite set $\mathcal{S}$ of n positive integers $s_1,s_2,\cdots,s_n$.
QUERY: Does there exist a subset $\mathcal{J}$ of the index set $\{1,2,\ldots,n\}$ of $\mathcal{S}$ such that

$$\sum_{i\in\mathcal{J}}s_i=\frac{1}{2}\sum_{i=1}^{n}s_i? \tag{3}$$

3. Complexity of the MRST

We examine the MRST first. Assume that the denominator in the objective function of $\mathcal{P}_1$ is unrestricted in sign. To study its complexity, we express MINIMUM RATIO SPANNING TREE as the following optimization problem:

INSTANCE: A graph $\mathcal{G}(\mathcal{V},\mathcal{A})$ with $|\mathcal{V}|=n$ nodes and $|A|=m$ arcs and m pairs of real numbers (a_i,b_i) with $a_i>0$ for each $i\in\mathcal{A}$ n nodes and $|A|$

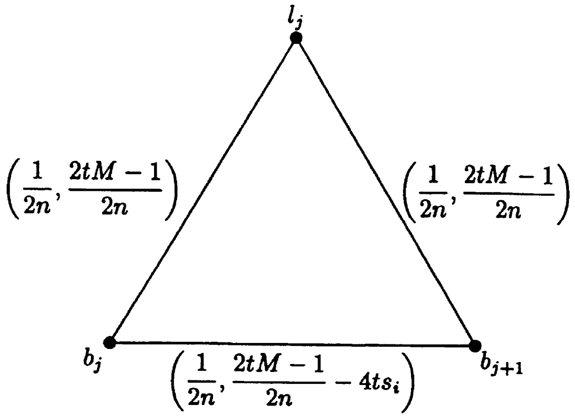

Figure 1. Basic building block for the constructed graph $\mathcal{G}$.

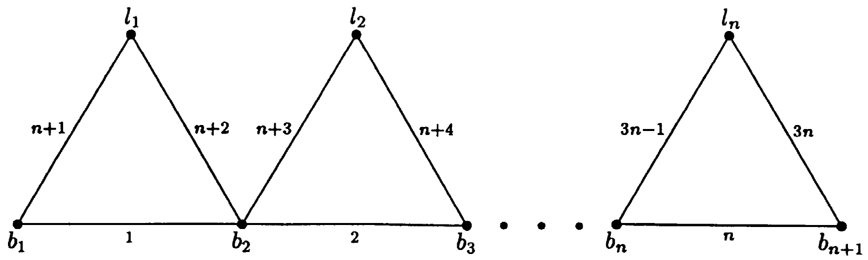

Figure 2. Constructed graph $\mathcal{G}$.

PROBLEM: Find a spanning tree such that

$$z = \min_{T \in \mathfrak{T}} \frac{\sum_{i \in T} a_i}{\sum_{i \in T} b_i} \tag{4}$$

THEOREM 1. *MINIMUM RATIO SPANNING TREE is NP–hard.*

Proof. We shall use a reduction from the NP–complete problem BIPARTITION following along the lines of [6]. Let $s_1, s_2, \ldots, s_n$ be any instance of BIPARTITION. We construct the following instance of MINIMUM RATIO SPANNING TREE: Let $M = \sum_{i=1}^{n} s_i$ so that the value of a feasible bipartition is $\frac{1}{2}M$. Let t be a positive integer. Define a graph $\mathcal{G}(\mathcal{V}, \mathcal{A})$ with $|\mathcal{V}| = 2n+1$ nodes and $|\mathcal{A}| = 3n$ arcs. The basic building block of our graph is the triangle shown in Figure 1.

Each arc carries a pair of real numbers (a_i, b_i) formed from the instance of BIPARTITION and the constants t and M. The graph $\mathcal{G}$ is constructed by stringing together the triangles along the base arcs as shown in Figure 2. The arcs are numbered from 1 to n, (left to right) along the base arcs so that the values s_i are in one-to-one correspondence with indices of the base arcs. The arcs representing the legs of each triangle are numbered from $n+1$ to $3n$, (left-to-right) across the set of triangles.

The base arcs of the triangle, denoted by the index set $\mathfrak{B}=\{1,\ldots,n\}$, have values

$$a_i = \frac{1}{2n} \tag{5}$$

$$b_i = \frac{2tM-1}{2n} - 4ts_i \tag{6}$$

The legs of the triangle are denoted by the index set $\mathfrak{L}=\{n+1,\ldots,3n\}$ and have values

$$a_i = \frac{1}{2n} \tag{7}$$

$$b_i = \frac{2tM-1}{2n}. \tag{8}$$

It should be clear that we can select any subset of the s_i by traversing the legs of the triangles and including arcs from $\mathfrak{B}$ as needed, and that any spanning tree must include two arcs from each triangle making up $\mathcal{G}$. Therefore, every spanning tree will contain a subset of arcs $\mathfrak{B}' \subseteq \mathfrak{B}$ and a subset of arcs $\mathfrak{L}' \subset \mathfrak{L}$. The set $\mathfrak{L}'$ always contains at least n arcs and we must have $|\mathfrak{B}' \cup \mathfrak{L}'| = 2n$.

We claim there exists a subset $\mathcal{J}$ of the index set $\{1,2,\ldots,n\}$ corresponding to a bipartition if and only if we can find a spanning tree $\mathcal{T}$ that minimizes

$$z = \frac{\sum_{i\in\mathcal{T}} a_i}{\sum_{i\in\mathcal{T}} b_i} \quad \forall \mathcal{T} \in \mathfrak{T}. \tag{9}$$

For any spanning tree $\mathcal{T} \in \mathfrak{T}$ on our constructed graph, the numerator of Eq. (9) is equal to 1. The value of the denominator is

$$D = \sum_{i\in\mathcal{L}'} \frac{2tM-1}{2n} + \sum_{i\in\mathfrak{B}'} \left(\frac{2tM-1}{2n} - 4ts_i\right) \qquad = 2t\sum_{i=1}^{n} s_i - 4t\sum_{i\in\mathfrak{B}'} s_i - 1. \tag{10}$$

Setting $\mathfrak{B}' = \mathcal{J}$ satisfies Eq. (3) with $z^* = -1$. If the denominator is less than -1 then $z > -1$ and if it is positive $z > 0$; as it is odd, it can never be equal to 0. Thus, we can recognize BIPARTITION if and only if we can solve the optimization problem MINIMUM RATIO SPANNING TREE.

4. Complexity of the TRMST

Now consider the TRMST problem. We express TWO RATIO MINIMUM SPANNING TREE as the following optimization problem:

INSTANCE: A graph $\mathcal{G}(\mathcal{V},\mathcal{A})$ with $|\mathcal{V}| = n$ nodes and $|\mathcal{A}| = m$ arcs and a set of 4 real numbers (a_i, b_i, c_i, d_i) all positive, for each $i \in \mathcal{A}$.

PROBLEM: Find a spanning tree $\mathcal{T} \in \mathfrak{T}$ such that

$$z = \min_{\mathcal{T} \in \mathfrak{T}} \left[\frac{\sum_{i \in \mathcal{T}} a_i}{\sum_{i \in \mathcal{T}} b_i} + \frac{\sum_{i \in \mathcal{T}} c_i}{\sum_{i \in \mathcal{T}} d_i} \right]. \tag{11}$$

THEOREM 2. *TWO RATIO MINIMUM SPANNING TREE is NP-hard.*

Proof. We shall use a reduction from BIPARTITION using the same construction of $\mathcal{G}$ as in the previous theorem. Let $s_1, s_2, \ldots, s_n$ be any instance of BIPARTITION. We construct the following instance of TWO RATIO MINIMUM SPANNING TREE: Let $M = \sum_{i=1}^{n} s_i$ so that the value of a feasible bipartition is $\frac{1}{2}M$. Let k and B be positive integers. Each arc carries 4 positive real numbers (a_i, b_i) and (c_i, d_i) formed from the instance of BIPARTITION and the constants k and B. The base arcs of the triangle are denoted by the index set $\mathfrak{B} = \{1, \ldots, n\}$ and have values

$$a_i = B, \tag{12}$$

$$b_i = 2B \frac{k + 2ns_i}{2k + M}, \tag{13}$$

$$c_i = 2B \frac{k + 2ns_i}{2k + M}, \tag{14}$$

$$d_i = B. \tag{15}$$

The legs of the triangle correspond to the index set $\mathcal{L} = \{n+1, \ldots, 3n\}$ with values

$$a_i = B, \tag{16}$$

$$b_i = 2B \frac{k}{2k + M}, \tag{17}$$

$$c_i = 2B \frac{k}{2k + M}, \tag{18}$$

$$d_i = B. \tag{19}$$

The basic building block of the graph $\mathcal{G}$ is shown in Figure 3 with the construction as shown in Figure 2.

We claim there exists a subset $\mathcal{J}$ of the index set $\{1, 2, \ldots, n\}$ satisfying Eq. (3) if and only if we can find a spanning tree $\mathcal{T} \in \mathfrak{T}$ such that

$$z = \min_{\mathcal{T} \in \mathfrak{T}} \frac{\sum_{i \in \mathcal{T}} a_i}{\sum_{i \in \mathcal{T}} b_i} + \frac{\sum_{i \in \mathcal{T}} c_i}{\sum_{i \in \mathcal{T}} d_i} \tag{20}$$

Since the two ratios are reciprocals of one another, we need only examine r_1. The value of the first ratio's numerator for any spanning tree $\mathcal{T} \in \mathfrak{T}$ is

$$\sum_{i \in \mathfrak{B}' \cup \mathfrak{L}'} B = 2Bn \tag{21}$$

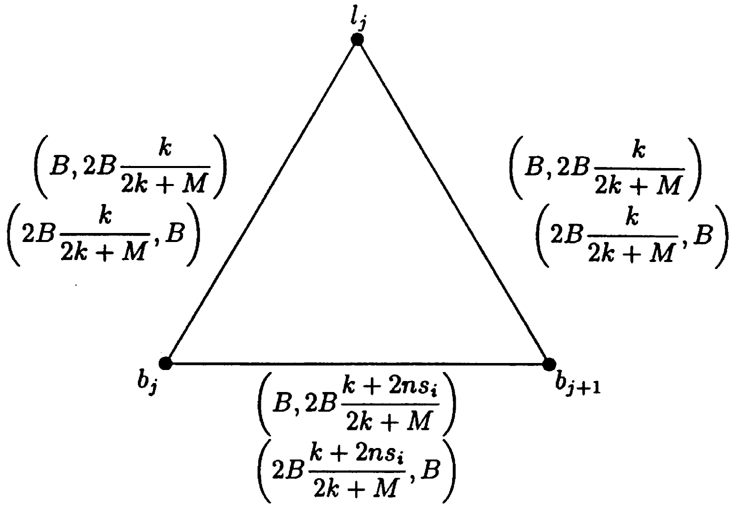

Figure 3. Building block for the graph $\mathcal{G}$.

since $|\mathfrak{B}' \cup \mathfrak{L}'| = 2n$. The value of the denominator for r_1 is

$$\sum_{i\in\mathfrak{B}'} 2B\frac{k+2ns_i}{2k+M} + \sum_{i\in\mathfrak{L}'} 2B\frac{k}{2k+M} \tag{22}$$

$$= \sum_{i\in\mathfrak{B}'} 2B\frac{2ns_i}{2k+M} + \sum_{i\in\mathfrak{B}'\cup\mathfrak{L}'} 2B\frac{k}{2k+M} \tag{23}$$

$$= 4Bn\left(\frac{\sum_{i\in\mathfrak{B}'} s_i}{2k+M} + \frac{k}{2k+M}\right) \tag{24}$$

$$= 4Bn\left(\frac{k+\sum_{i\in\mathfrak{B}'} s_i}{2k+M}\right). \tag{25}$$

Taking account of the numerator, we have

$$r_1 = \frac{2k+M}{2(k+\sum_{i\in\mathfrak{B}'} s_i)} \tag{26}$$

$$= \frac{k+\frac{1}{2}M}{k+\sum_{i\in\mathfrak{B}'} s_i}. \tag{27}$$

Therefore,

$$z = \frac{k+\frac{1}{2}M}{k+\sum_{i\in\mathfrak{B}'} s_i} + \frac{k+\sum_{i\in\mathfrak{B}'} s_i}{k+\frac{1}{2}M} \tag{28}$$

is the value of any spanning tree $\mathcal{T} \in \mathfrak{T}$, depending on our choice of $\mathfrak{B}'$. Choosing $\mathfrak{B}' = \mathcal{J}$, a bipartition, yields the value $z^* = 2$. We claim this to be the minimum value of any spanning tree on our graph $\mathcal{G}$. We therefore need to show that any other spanning tree on $\mathcal{G}$ gives a value $z > 2$.

Suppose we choose $\mathfrak{B}' \subseteq \mathfrak{B}$ such that $\sum_{i\in\mathfrak{B}'} s_i > \frac{1}{2}M$. That is, $\sum_{i\in\mathfrak{B}'} s_i = \frac{1}{2}M + l$ for some $0 < l \leqslant \frac{1}{2}M$. We can then express Eq. (28) as

$$z = \frac{k+\frac{1}{2}M}{k+\frac{1}{2}M+l} + \frac{k+\frac{1}{2}M+l}{k+\frac{1}{2}M}. \tag{29}$$

The sum of the two ratios in this case is

$$z = 2 + \frac{2l}{2k+M} - \frac{2l}{2k+M+2l} \tag{30}$$

and so $z > 2$. Choosing $\sum_{i\in\mathfrak{B}'} s_i < \frac{1}{2}M$, we have

$$z = \frac{k+\frac{1}{2}M}{k+\frac{1}{2}M-l} + \frac{k+\frac{1}{2}M-l}{k+\frac{1}{2}M} \tag{31}$$

$$= 2 + \frac{2l}{2k+M-2l} - \frac{2l}{2k+M} \tag{32}$$

and $z > 2$ in this case as well. Hence, we can recognize BIPARTITION if and only if we can solve the optimization problem TWO RATIO MINIMUM SPANNING TREE.

5. Optimality Considerations

Necessary and sufficient conditions for an optimal solution to the MRST are well known [2]. Assuming the denominator is always positive, the following parametric version of the problem is used in algorithmic solutions:

$$\min_{\mathcal{T}\in\mathfrak{T}} \left[\sum_{i\in\mathcal{T}} a_i - \delta \sum_{i\in\mathcal{T}} b_i \right]. \tag{33}$$

Now, pick a real number $\bar{\delta}$ and evaluate

$$\mathcal{H}(\bar{\delta}) = \min_{\mathcal{T}\in\mathfrak{T}} \left[\sum_{i\in\mathcal{T}} a_i - \bar{\delta} \sum_{i\in\mathcal{T}} b_i \right]. \tag{34}$$

Finding the MST with respect to these new cost values results in some spanning tree $\bar{\mathcal{T}}$. If $\mathcal{H}(\bar{\delta}) = 0$, then $z^* = \bar{\delta}$ is the optimal value of the MRST with the optimal tree being $\bar{\mathcal{T}}$. This general procedure, proposed by Dinklebach [3], applies to quasi–convex fractional programs.

An attempt was made to extend this result to problems involving more than one ratio [1]. Indeed, it seems natural to extend the parametric equation to multiple ratios. In the context of the TRMST, this results in

$$H(\delta) = \min_{\mathcal{T}\in\mathfrak{T}} \left[\left(\sum_{i\in\mathcal{T}} a_i - \delta_1 \sum_{i\in\mathcal{T}} b_i \right) + \left(\sum_{i\in\mathcal{T}} c_i - \delta_2 \sum_{i\in\mathcal{T}} d_i \right) \right] \tag{35}$$

where $\delta = (\delta_1, \delta_2)$ are the values of the two ratios. A necessary and sufficient condition was put forth in [1] claiming that δ^* was the optimal solution to Eq. (35) with an optimal spanning tree $\mathcal{T}^*$ if and only if δ^* solved $\mathcal{H}(\delta^*) = 0$. A direct counter-example for linear programs, presented in [5] showed this was not necessarily true. A sufficient condition for an optimal solution to the problem of optimizing a sum of ratios of two linear functions was subsequently established in [5].

In the context of our problem, we attempt to show why the postulated necessary and sufficient conditions of [1] fail. We also attempt to provide some insight into why the TRMST is such a hard problem. For completeness, we reprise some of the results of [5].

5.1. THE IMAGE SPACE

Imagine we enumerated all the spanning trees on $\mathcal{G}$ and recorded the values for each ratio forming a set $\mathcal{R}$. Expressed formally, we have

$$\mathcal{R} = \left\{ (r_1, r_2) | r_1 = \frac{\sum_{i \in \mathcal{T}} a_i}{\sum_{i \in \mathcal{T}} b_i}, r_2 = \frac{\sum_{i \in \mathcal{T}} c_i}{\sum_{i \in \mathcal{T}} d_i}; \forall \mathcal{T} \in \mathfrak{T} \right\}. \tag{36}$$

So each spanning tree $\mathcal{T} \in \mathfrak{T}$ maps non-uniquely to a point $r = (r_1, r_2)$. By plotting r_1 against r_2, we have a solution space in which the linear objective function $r_1 + r_2$ can be minimized. Initial upper and lower bounds can be easily identified and using these, a triangle in $\mathcal{R}$-space can be constructed that contains the optimal solution. Figure 4 illustrates these ideas.

Easily, the point $u = (\min r_1, \min r_2)$ provides a lower bound on the optimal solution. Now evaluate each of these points to get the value of the other ratio. This produces two other points $l = (l_1, l_2)$ and $\hat{v} = (\hat{v}_1, \hat{v}_2)$, and bounds a region guaranteed to contain the optimal solution. This is shown as the shaded region in Figure 4.

We also get two upper bounds–the iso-contour value passing through the points l and $\tilde{v}$ and one passing through the points l and v, the latter point being determined by intersection. We pick the smaller of these as the initial upper bound. This is labeled f_{up}^0.

The basis for the Image Space algorithm is now in place. Following our example, minimizing the second ratio while imposing the constraint $r_1 \leqslant v$ results in an improved upper and lower bound (e.g. f_{up}^1 as shown in Figure 4). In effect, the algorithm seeks to continually shrink the triangle by iteratively descending along each axis of $\mathcal{R}$ (*cf.* [7] for the algorithmic details).

5.2. A SUFFICIENT CONDITION

A sufficient condition for the sum of ratios programming problem was set forth in [5] using the parametric function $\mathcal{H}(\delta)$ and its properties with the image space $\mathcal{R}$.

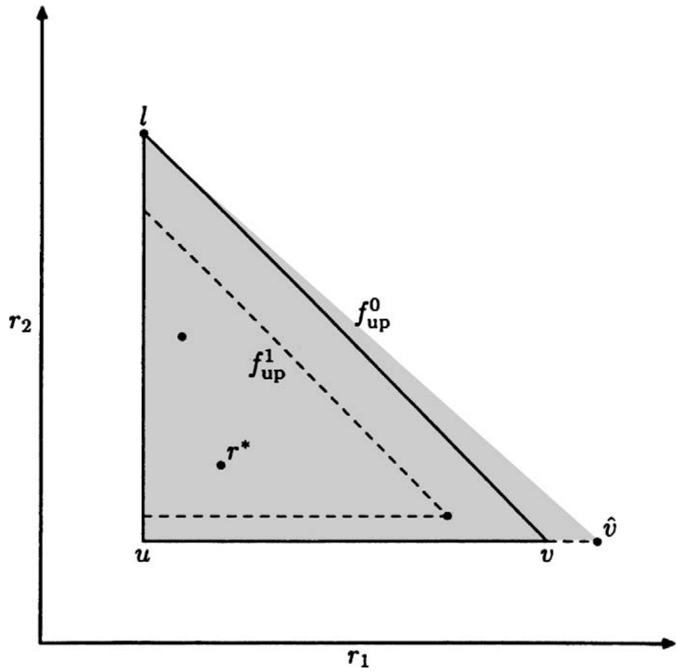

Figure 4. Image space representation for the sum of ratios optimization problem. The shaded region spans extreme points of $\mathcal{R}$ containing the optimal solution. Superscripts index successive iterations of the Image Space algorithm.

We now state Falk and Palocsay's sufficient condition for a globally optimal solution to $\mathcal{P}_2$. In what follows, let $n_i(\mathcal{T}), d_i(\mathcal{T})$ denote the linear functions associated with the numerator and denominator of the ith ratio $(i = l, \ldots, m)$ for some $\mathcal{T} \in \mathfrak{T}$, respectively.

THEOREM 3. (*Falk and Palocsay*). *Let u, l and v be the extreme points of a triangular region in $\mathcal{R}$ determined by upper and lower bounds as described in Section* 5.1 *with l being a feasible solution to $\mathcal{P}_2$. Let $\mathcal{H}$ be as defined in Section* 5.3 *with parameters u, l and v. If $\mathcal{H}(l) = 0$, $\mathcal{H}(u) > 0$ then $z^* = l_1 + l_2$ is the optimal objective function value for $\mathcal{P}_2$ and the spanning tree $\mathcal{T}^*$ corresponding to l is an optimal solution to $\mathcal{P}_2$.*

Turning back to the construction in Theorem 2, the optimal solution is the point $\delta = (1, 1)$ corresponding to a bipartition. Applying the definition of $\mathcal{H}(\delta)^*$ with

$\delta^* = (1,1)$, we find

$$\mathcal{H}(\delta)^* = \min_{\mathcal{T}\in\mathfrak{T}}\left[\left(\sum_{i\in\mathcal{T}} a_i - \delta_i^* \sum_{i\in\mathcal{T}} b_i\right) + \left(\sum_{i\in\mathcal{T}} c_i - \delta_2^* \sum_{i\in\mathcal{T}} d_i\right)\right] \tag{37}$$

$$= k + \frac{1}{2}M - \left(k + \sum_{i\in\mathfrak{V}_i} s_i\right) + \left(k + \sum_{i\in\mathfrak{V}_i} s_i\right) - \left(k + \frac{1}{2}M\right) \tag{38}$$

$$= 0 \tag{39}$$

for any constructed spanning tree. The Almogy–Levin theorem again fails rather spectacularly. Yet, it points up the difficulty of this class of problems.

Where could they have gone wrong? We attempt to provide some resolution to this question in the context of the image space approach. Quite nicely, this yields some insight into why the problem at hand is so hard. From [5], we have the following properties of $\mathcal{H}(r)$.

THEOREM 4. *$\mathcal{H}(r)$ is (piecewise) concave over $\Re^m$.*

THEOREM 5. *$\mathcal{H}(r) \leqslant 0$ for all $r \in \mathcal{R}$.*

THEOREM 6. *$\mathcal{H}(r) < 0$ for every r such that $r_i > \frac{n_i(\mathcal{T}^*)}{d_i(\mathcal{T}^*)}$ for all i.*

Given these results, it is tempting to conclude that $\mathcal{H}(r) = 0$ is a necessary and sufficient condition for an optimal solution. As we now know, it is only sufficient. The following, due to [4], characterizes a property of a subset of the points $r \in \mathcal{R}$.

DEFINITION 7. *A point r is pareto–optimal if $\tilde{r} \leqslant r$ and $\tilde{r} \in \mathcal{R} \rightarrow \tilde{r} = r$.*

In the discrete case, $\mathcal{R}$ is just a collection of points so this definition defines the convex hull of $\mathcal{R}$ contained within the shaded region of Figure 4. In general, Definition 7 describes the boundary of $\mathcal{R}$ where we will find the optimal solution whether or not $\mathcal{R}$ is convex.

The next result characterizes $\mathcal{H}(r)$ for points that do *not* conform to Definition 7. Although we express it in problem–specific terms, its general nature should be quite obvious.

THEOREM 8. *$\mathcal{H}(r) < 0$ for any point $r \in \mathcal{R}$ that is not pareto-optimal.*

Proof. If r is not pareto-optimal, then there exists a point $\tilde{r} \in \mathcal{R}$ such that $\tilde{r} \leqslant r$ but $\tilde{r} \neq r$. So there is a j such that $\tilde{r}_j < r_j$ but $\tilde{r}_i < r_i$ for all i. Consequently, there

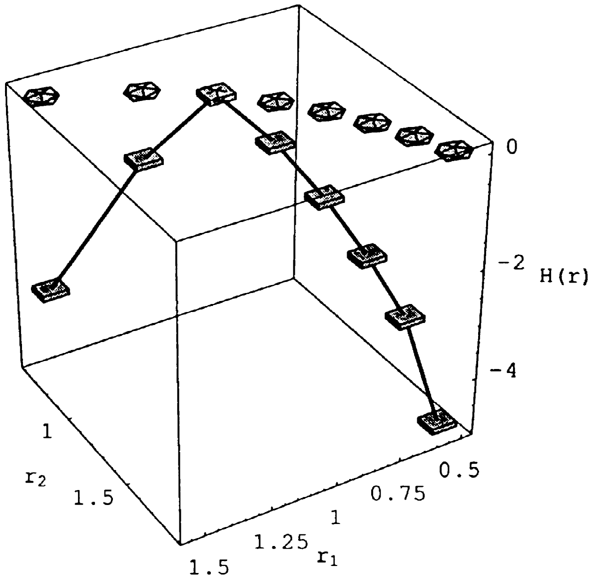

Figure 5. Image space for TRMST corresponding to an instance of bipartition.

is an $\tilde{\mathcal{T}} \in \mathfrak{T}$ such that $\frac{n_i(\tilde{\mathcal{T}})}{d_i(\tilde{\mathcal{T}})}$ for all i and $\frac{n_j(\tilde{\mathcal{T}})}{d_j(\tilde{\mathcal{T}})} < r_j$. Therefore,

$$\mathcal{H}(r) = \min_{\mathcal{T}\in\mathfrak{T}} \sum_{i=1}^{m} [n_i(\mathcal{T}) - r_i d_i(\mathcal{T})] \tag{40}$$

$$\leqslant \sum_{i=1}^{m} [n_i(\tilde{\mathcal{T}}) - r_i d_i(\tilde{\mathcal{T}})] \tag{41}$$

$$< 0 \tag{42}$$

Let's consider this result in the light of Theorem 2. For the TRMST instance of Theorem 2, it is not difficult to see that every $r \in \mathcal{R}$ is pareto-optimal. Suppose we have the points $\tilde{r} = (\tilde{r}_1, \tilde{r}_2)$ with $\tilde{r} \in \mathcal{R}$ and $r = (r_1, r_2)$, $r \in \mathcal{R}$. If $\tilde{r}_1 \leqslant r_1$ then $\tilde{r}_2 \geqslant r_2$ because $\tilde{r}_1 = \frac{1}{\tilde{r}_2}$. Therefore, $\tilde{r} \leqslant r \Rightarrow \tilde{r} = r$.

As an example, consider a TRMST instance of Theorem 2 consisting of the set $S = (1, 2, 3, 4)$ with $k = 1$ (note that B drops out of the picture). Figure 5 shows $\mathcal{R}$ and the (piecewise) concave function $\mathcal{H}(r)$ with $\mathcal{H}(r) = 0$ at the optimal point $r^* = (1, 1)$.

This illustrates part of what makes this problems so hard. Any smart algorithm would want to only consider pareto–optimal points as solution candidates and these points are exponential in number. As observed in [7], the Image Space algorithm tends to trace out the so-called *pareto–optimal* points. Clever heuristics can be used to find a "good" solution (an upper bound) and so reduce the overall computational effort, but in general, this is a tough problem to solve.

6. Summary

In addition to demonstrating the complexity of these problems, we hope to have shed some light into what makes the two-–ratio version of the MRST so difficult. The efficacy of the sufficient condition, we hope, is now a bit clearer as well.

For pareto–optimal points, the only conclusion is that $\mathcal{H}(r) \leqslant 0$ and for all else $\mathcal{H}(r) < 0$. So perhaps this is where Almogy and Levin went wrong. They must have recognized Theorem 8 and succumbed to the conclusion that $\mathcal{H}(r) = 0$ for the optimal point when in fact, it can be true for any pareto–optimal point.

The Falk–Palocsay sufficient condition and our analysis points out the now obvious fact that the only candidates for an optimal solution are those lying along the appropriate boundary of $\mathcal{R}$. For the TRMST, our reduction to the bipartition problem shows that the number of such points can be exponentially large.

7. Acknowledgements

Our thanks to Mark Shelor for his insightful comments on earlier drafts of this paper.

References

Almogy, Y. and Levin, O. (1971), A class of fractional programming problems, *Operations Research*, 1957–67.

Chandrasekaran, R. (1977), Minimal ratio spanning trees, *Networks* 7, 355–342.

Dinklebach, W. (1967), On nonlinear fractional programming, *Management Science*, 13, 492–498.

Falk, J.E. Private communication.

Falk, J.E. and Palocsay, S.W. (1992), Optimizing the sum of linear fractional functions. In: *Recent Advances in Global Optimization*, Floudas, C. and Pardalos, M. (eds), Princeton University Press, Princeton, NJ, pp. 221–258.

Hansen, P., De Aragão, M.V.P. and Ribeiro, C.C. (1991), Hyperbolic 0-1 programming and query optimization in information retrieval, *Mathematical Programming* 52, 255–263.

Skiscim, C.C. and Palocsay, S. (2001), Minimum spanning trees with sums of ratios, *Journal of Global Optimization*, 19, 103–120.

Note added in proof

Professor Francesco Maffoli informs us that his work published in the 5th Conference on Optimization Techniques, Part 2, pp. 110–117, *Lecture Notes in Computer Science*, 4, Springer, 1973, R. Conti, Antonis Ruberti (Eds.), was the first to propose a polynomial-time solution to the Minimum Ratio Spanning Tree Problem.